

Question Prove that the n th roots of unity form a multiplicative group.

Proof: Let us suppose that S be the set of n th roots of unity,

i.e., $S = \{e^{2\pi i r/n} : r=0,1,2,3,\dots,(n-1)\}$ where $i = \sqrt{-1}$.

Let a and $b \in S$ given by

$$a = e^{2\pi i r_1/n}, r_1 \leq n-1$$

$$\text{and } b = e^{2\pi i r_2/n}, r_2 \leq n-1$$

Now we are going to show first of all that $ab \in S$.

$$\text{Now } ab = e^{2(r_1+r_2)\pi i/n} \quad (\text{By adding } a \text{ & } b \text{ from above})$$

If $r_1+r_2 \leq n-1$, then obviously $ab \in S$

If $r_1+r_2 > n-1$, then let $r_1+r_2 = n+k$ so that $k \leq n-2$

$$[\text{The greatest value of } r_1+r_2 = (n-1)+(n-1) = 2n-2]$$

Therefore, it follows from $r_1+r_2 = n+k$ that the greatest value of $k = (r_1+r_2)-n = 2n-2-n = n-2$

$$\text{In this case, } ab = e^{2(n+k)\pi i/n}$$

$$= e^{2\pi i} e^{2k\pi i/n} \in S, \quad \because k \leq n-2$$

Thus we have in both the cases $ab \in S$

Hence the first postulate is satisfied.

Associativity:- Since the elements of S are complex numbers and the multiplication of complex numbers is associative.

Hence the second axiom is satisfied.

Existence of Identity : The identity of S is $e^{2 \cdot 0 \pi i/n} = 1$

∴ existence of identity is satisfied. Since $r=0$

Existence of inverse : The inverse of any element $e^{2\pi i r/n}$ is $e^{2(n-r)\pi i/n}$, for their product

we have,

$$- e^{2\pi i r/n} \cdot e^{2(n-r)\pi i/n} = e^{2\pi i r/n} + e^{2\pi i n} - \frac{2\pi i r}{n} = e^{2\pi i} = 1$$

∴ existence of inverse is satisfied

Hence finally the four postulates are satisfied.

Thus the set S forms a group.

Moreover, the multiplication of two complex numbers is ~~not~~ commutative.

Hence S is an Abelian group.

Hence the result